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TYPE AND INFRATYPE IN SYMMETRIC SEQUENCE SPACES

BY

MICHEL TALAGRAND*

Institut de Mathématiques, Université Paris VI
4 pl. Jussieu, 75230 Paris Cedex 05, France
and
Department of Mathematics, The Ohio State University
231 West 18th Avenue, Columbus, OH 43210, USA
e-mail: Michel@Talagrand.net

Dedicated to J. Lindenstrauss

ABSTRACT

We construct a symmetric sequence space that is of infratype 2 but not of type 2.

1. Introduction

A Banach space X is said to be of infratype p if there exists a number C such that for any $N > 0$ and any vectors $x_1, \ldots, x_N$ of X, one can find signs $\eta_j \in \{-1, 1\}$ such that

$$\Big\| \sum_{j \le N} \eta_j x_j \Big\| \le C \Big(\sum_{j \le N} \|x_j\|^p \Big)^{1/p}. \tag{1.1}$$

A Banach space X is said to be of type p if there exists a number C such that for any $N > 0$ and any vectors $x_1, \ldots, x_N$ of X, one has

$$\mathrm{Av} \Big\| \sum_{j \le N} \eta_j x_j \Big\| \le C \Big(\sum_{j \le N} \|x_j\|^p \Big)^{1/p}, \tag{1.2}$$

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where Av denotes averaging over all choices of signs. Obviously, if a Banach space is of type p it is also of infratype p. For $p < 2$, the converse is proved in [2]. This paper will prove that this converse does not hold for $p = 2$.

THEOREM 1.1: *There exists a symmetric sequence space X that is of infratype 2 but not of type 2.*

Let us recall that a symmetric sequence space is a space that admits a 1-unconditional basis, and such that the norm of a vector is invariant under permutation of its coordinates in this basis.

A Banach space X is said to be of sup-cotype q if there exist a number C such that for any $N > 0$ and any vectors $x_1, \dots, x_N$ of X we can find signs η_i such that

$$\left(\sum_{j\le N} \|x_j\|^q\right)^{1/q} \le C \left\| \sum_{j \le N} \eta_j x_j \right\|, \tag{1.3}$$

and it is said to be of cotype q if (1.3) can be replaced by

$$\left(\sum_{j\le N} \|x_j\|^q\right)^{1/q} \le C \operatorname{Av} \left\| \sum_{j \le N} \eta_j x_j \right\|. \tag{1.4}$$

It is proved in [3] that for $q > 2$, a Banach space X is of cotype q if and only if it is of sup-cotype q. For $q = 2$, this is not true, even if X is a symmetric sequence space, as is shown in [4].

When trying to construct a symmetric sequence space X of infratype 2 but not of type 2, the first approach that comes to mind is that the dual of the space of [4] should be a good candidate. However, the symmetry between (1.1) and (1.3) is rather formal. Given vectors $(x_j)_{j\le N}$, it is much easier to find signs $(\eta_j)_{i\le N}$ such that $\|\sum_{j\le N} \eta_j x_j\|$ is large than to find signs such that $\|\sum_{j\le N} \eta_j x_j\|$ is small. There is much less room to construct the example that proves Theorem 1.1 than the space of [4]. Not surprisingly, however, the dual of the example we are about to construct is a new example of a space that is of sup-cotype 2 but not of cotype 2.

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2. The construction; basics properties

The construction depends on a sequence $(m_k)_{k\geq 1}$ of integers, and on a parameter $r > 2$. The space X will be of cotype r.

We observe that X cannot be of cotype 2, or even of sup-cotype 2. Indeed, in that case, if (e_i) denotes the canonical basis of X, assuming without loss of generality that $\|e_i\| = 1$, we have $\|\sum_{i\leq N} \eta_i a_i e_i\| = \|\sum_{i\leq N} a_i e_i\|$ by unconditionality, so that

$$\Big\|\sum_{i\leq N} a_i e_i\Big\| \geq \frac{1}{C}\Big(\sum_{i\leq N} a_i^2\Big)^{1/2}$$

if X is of sup-cotype 2, and

$$\Big\|\sum_{i\leq N} a_i e_i\Big\| \leq C\Big(\sum_{i\leq N} a_i^2\Big)^{1/2}$$

if X is of infratype 2. Thus in that case X is isomorphic to a Hilbert space, and hence of type 2 and of cotype 2.

The only requirement for the sequence $(m_k)_{k\geq 1}$ is that it increases fast enough; how fast is required for our construction will be determined later. We set $n_k = 2km_k$, and we already assume that $n_k \geq n_{k-1}^{2(k+1)}$.

Consider the space Z of functions $\mathbb{N} \to \mathbb{R}$ with finite support. The space X will be the completion of Z for a certain norm. For x, y in Z we write

$$\langle y, x\rangle = \sum_{i\geq 1} y(i)x(i).$$

The bracket $\langle\cdot\rangle$ will also be used to denote the duality between X and its dual X^*.

To lighten notation, for $x \in Z$ we write

$$\mathrm{CS}(x) = \mathrm{card}\{i \in \mathbb{N}; x(i) \neq 0\},$$

the cardinality of the support of x.

Given $a > 0$, and integers $s < t$, we consider the norm

$$\|x\|_{a,s,t} = \sup\Big\{\langle y, x\rangle \colon \forall i, |y(i)| \leq \frac{a}{\sqrt{s}}, \|y\|_2 \leq a; \mathrm{CS}(y) \leq t\Big\}, \tag{2.1}$$

where $\|\cdot\|_2$ denotes the ℓ^2 norm. For $k \geq 1$ and $0 \leq \ell \leq k$ we consider the norm

$$\|x\|_{k,\ell} = \|x\|_{a,s,t} \quad \text{where } a = 2^{-k+\ell}, s = n_k^\ell \text{ and } t = n_k^{\ell+1}.$$

We consider the norms

$$\||x\||_1 = \Big(\sum_{k\geq 1, 0\leq \ell\leq k} \|x\|_{k,\ell}^r\Big)^{1/r}, \tag{2.2}$$

$$\||x\||_2 = \Big(\sum_{k \text{ even}} \Big(\sum_{0\leq \ell\leq k} \|x\|_{k,\ell}^r\Big)^{2/r}\Big)^{1/2}, \tag{2.3}$$

$$\||x\||_3 = \Big(\sum_{k \text{ odd}} \Big(\sum_{0\leq \ell\leq k} \|x\|_{k,\ell}^r\Big)^{2/r}\Big)^{1/2}, \tag{2.4}$$

and finally

$$\|x\| = \inf\{\||x_1\||_1 + \||x_2\||_2 + \||x_3\||_3; x = x_1 + x_2 + x_3\}. \tag{2.5}$$

In (2.3) and everywhere, "k even" means "k even and $k \geq 2$".

The space X is the completion of $(Z, \|\cdot\|)$, so it is obviously a symmetric sequence space. Of course, the ideas behind these definitions will become clear only gradually. We first have to learn how to work with the norms $\|\cdot\|_{k,\ell}$.

We start with simple facts.

LEMMA 2.1: *We have*

$$\|x\|_{a,s,t} \leq a\|x\|_2, \tag{2.6}$$

$$\|x\|_{a,s,t} \leq a\sqrt{t}\|x\|_\infty, \tag{2.7}$$

$$\|x\|_{a,s,t} \leq \|x\|_2\sqrt{\mathrm{CS}(x)}\frac{a}{\sqrt{s}}. \tag{2.8}$$

Of course here $\|\cdot\|_\infty$ denotes the supremum norm.

Proof: To prove (2.6) we use that $\langle y, x\rangle \leq \|y\|_2\|x\|_2$. To prove (2.7) we use that $\langle y, x\rangle \leq \|y\|_1\|x\|_\infty \leq \|y\|_2\sqrt{\mathrm{CS}(y)}\|x\|_\infty$, and to prove (2.8) we use that $\langle y, x\rangle \leq \|y\|_\infty\|x\|_1 \leq \|y\|_\infty\|x\|_2\sqrt{\mathrm{CS}(x)}$. ∎

LEMMA 2.2: *If $\|x\|_{a,s,t} \leq b$ and if the sequence $(|x(i)|)$ is non-increasing, then*

$$|x(s)| \leq \frac{b}{a\sqrt{s}}, \tag{2.9}$$

$$\sum_{s\leq i<s+t} x(i)^2 \leq \frac{b^2}{a^2}. \tag{2.10}$$

Proof: By taking $y(i) = \pm a/\sqrt{s}$ for $i \leq s$ $(< t)$ and $y(i) = 0$ if $i > s$, we get

$$s|x(s)| \leq \sum_{i\leq s} |x(i)| \leq \frac{b\sqrt{s}}{a}$$

and this proves (2.9).

For $s \le i < s+t$, let $y(i) = x(i)/Ab$, where

$$A = \max\left(\frac{1}{a^2}, \frac{1}{ab}\Big(\sum_{s \le i < s+t} x(i)^2\Big)^{1/2}\right),$$

and let $y_i(0) = 0$ if $i < s$ or $i \ge s+t$.

Thus, by (2.9), and since $|x(i)| \le |x(s)|$ for $i \ge s$, we have $|y(i)| \le a/\sqrt{s}$, and we also have $\sum_{s \le i < s+t} y(i)^2 \le a^2$.

Thus

$$\langle y, x\rangle = \frac{1}{Ab} \sum_{s \le i < s+t} x^2(i) \le b$$

and this implies (2.10). ∎

We define $c_0 = c_1 = 1$, and for $k \ge 2$ we define

$$c_k = 2^k / \sqrt{n_{k-1}^k}. \tag{2.11}$$

We define $c_{k,0} = c_k$ and for $1 \le \ell \le k$ we define

$$c_{k,\ell} = 2^{k-\ell} / n_k^{\ell/2}. \tag{2.12}$$

For secondary technical reasons, we will also use the norm $\mathcal{N}_k$ defined by $\mathcal{N}_k(x) = \|x\|_\infty$ for $k = -1, 0, 1$ and for $k \ge 2$ by

$$\mathcal{N}_k(x) = \sup\{\langle x, y\rangle; \|y\|_2 \le 1, \mathrm{CS}(y) \le n_{k-1}^k\}. \tag{2.13}$$

Thus for $k \ge 2$ we have

$$\|x\|_{k-1,k-1} \le \mathcal{N}_k(x). \tag{2.14}$$

LEMMA 2.3: *If $|||\, x_1 \,||| \le 1$, we can find a decomposition*

$$x = \sum_{k \ge 1, 0 \le \ell \le k} x_{k,\ell}$$

and we can find numbers $\alpha_{k,\ell} \ge 0$ such that the elements $x_{k,\ell}$ have disjoint supports and

$$\sum_{k \ge 1, 0 \le \ell \le k} \alpha_{k,\ell}^r \le L, \tag{2.15}$$

$$\|x_{k,\ell}\|_2 \le 2^{k-\ell}\alpha_{k,\ell}, \tag{2.16}$$

$$\|x_{k,\ell}\|_\infty \le c_{k,\ell}\alpha_{k,\ell}, \tag{2.17}$$

$$\mathrm{CS}(x_{k,\ell}) \le n_k^{\ell+1}, \tag{2.18}$$

$$\mathcal{N}_k(x_{k,0}) \le \alpha_{k,0}. \tag{2.19}$$

Here and below, L denotes a number that depends only on X, and that need not be the same at each occurrence.

Proof: Without loss of generality we assume that the sequence $(|x(i)|)$ is non-increasing. We set $\beta_{k,\ell} = \|x\|_{k,\ell}$, and if $\ell \geq 1$ we set $\alpha_{k,\ell} = \beta_{k,\ell}$. We set $\alpha_{1,0} = 2\beta_{1,0}$, and if $k \geq 2$ we set $\alpha_{k,0} = \beta_{k,0} + \beta_{k-1,k-1}$. Thus (2.15) holds by (2.2).

Setting $n_0 = 1$, for $k \geq 1$ we define $x_{k,0}$ by $x_{k,0}(i) = x(i)$ if $n_{k-1}^k \leq i < n_k$ and $x_{k,0}(i) = 0$ otherwise. We use (2.10) with $s = 1, t = n_k, a = 2^{-k}$ and $b = \beta_{k,k-1}$ to see that $\sum_{i \leq n_k} x(i)^2 \leq 2^{2k}\beta_{k,0}^2$, so that $\|x_{k,0}\|_2 \leq 2^k\beta_{k,0} \leq 2^k\alpha_{k,0}$. Moreover, for $i \geq n_{k-1}^k$ we have $ix(i)^2 \leq 2^{2k}\beta_{k,0}^2$ and thus $\|x_{k,0}\|_\infty \leq c_k\beta_{k,0} \leq c_k\alpha_{k,0}$. If $k \geq 2$ we use (2.10) with $s = n_{k-1}^{k-1}, t = n_{k-1}^k$, $a = 1$ and $b = \beta_{k-1,k-1}$ to see that $\sum_{n_{k-1}^{k-1} \leq i < n_{k-1}^k + n_{k-1}^{k-1}} x(i)^2 \leq \beta_{k-1,k-1}^2$, so that (since the sequence $(|x(i)|)$ is non-increasing) we have $\mathcal{N}_k(x_{k,k-1}) \leq \beta_{k-1,k-1} \leq \alpha_{k,0}$. If $k = 1$ we have $\mathcal{N}_1(x_{1,0}) = \|x_{1,0}\|_\infty \leq \|x_{1,0}\|_2 \leq 2\beta_{1,0} = 2\alpha_{1,0}$.

For $1 \leq \ell \leq k$, we define $x_{k,\ell}$ by $x_{k,\ell}(i) = x(i)$ if $n_k^\ell \leq i < n_k^{\ell+1}$ and $x_{k,\ell}(i) = 0$ otherwise. We observe that (2.18) is obvious by construction. We use (2.10) with $s = n_k^\ell, t = n_k^{\ell+1}$, and $a = 2^{-k+\ell}, b = \beta_{k,\ell}$ to prove (2.16) and (2.9) to get

$$\|x_{k,\ell}\|_\infty \leq \frac{2^{k-\ell}}{n_k^{\ell/2}}\alpha_{k,\ell} \leq c_{k,\ell}\alpha_{k,\ell}. \qquad \blacksquare$$

We need the following kind of converse to Lemma 2.3.

LEMMA 2.4: *Assume that for $k \geq 1$ and $0 \leq \ell \leq k$ we are given $x_{k,\ell} \in Z$ and $\alpha_{k,\ell} \geq 0$ such that*

$$\|x_{k,\ell}\|_2 \leq 2^{k-\ell}\alpha_{k,\ell}, \tag{2.20}$$

$$\|x_{k,\ell}\|_\infty \leq c_k\alpha_{k,\ell}, \tag{2.21}$$

$$\mathrm{CS}(x_{k,\ell}) \leq n_k^{\ell+2}, \tag{2.22}$$

$$\mathcal{N}_k(x_{k,\ell}) \leq 2^{-\ell}\alpha_{k,\ell}. \tag{2.23}$$

Then if $x = \sum_{k\geq 1, 0\leq \ell \leq k} x_{k,\ell}$ we have

$$|||\, x \,|||_1 \leq L\Big(\sum_{k\geq 1, 0\leq\ell\leq k} \alpha_{k,\ell}^r\Big)^{1/r}. \tag{2.24}$$

The reader observes the exponent $\ell + 2$ rather than $\ell + 1$ in (2.22) and the term c_k rather than $c_{k,\ell}$ in (2.21).

Proof: For any integers $k', \ell' \geq 0$, we have

$$\|x\|_{k',\ell'} \leq \sum_{k,\ell} \|x_{k,\ell}\|_{k',\ell'},$$

where to lighten notation we keep implicit that the summation is over $k \geq 1$ and $0 \leq \ell \leq k$. We are going to prove that

$$\|x_{k,\ell}\|_{k',\ell'} \leq \alpha_{k,\ell} a(k,\ell,k',\ell') \tag{2.25}$$

where

$$\forall k,\ell, \quad \sum_{k',\ell'} a(k,\ell,k',\ell') \leq L, \tag{2.26}$$

$$\forall k',\ell', \quad \sum_{k,\ell} a(k,\ell,k',\ell') \leq L. \tag{2.27}$$

The convexity of the function $t \mapsto t^r$ shows from (2.25) that

$$\|x\|^r_{k',\ell'} \leq \Big(\sum_{k,\ell} a(k,\ell,k',\ell')\Big)^{r-1} \sum_{k,\ell} \alpha^r_{k,\ell} a(k,\ell,k',\ell').$$

Summation over k', ℓ' and use of (2.26) and (2.27) then establish (2.24).

The rest of the proof is made tedious by the need to distinguish cases. From (2.6), (2.7) and (2.8) respectively we get that

$$\|x_{k,\ell}\|_{k',\ell'} \leq 2^{-k'+\ell'+k-\ell} \alpha_{k,\ell}, \tag{2.28}$$

$$\|x_{k,\ell}\|_{k',\ell'} \leq 2^{-k'+\ell'+k-\ell} \sqrt{n_{k'}^{\ell'+1}} c_k \alpha_{k,\ell}, \tag{2.29}$$

$$\|x_{k,\ell}\|_{k',\ell'} \leq 2^{-k'+\ell'+k-\ell} \sqrt{\frac{n_k^{\ell+2}}{n_{k'}^{\ell'}}} \alpha_{k,\ell}. \tag{2.30}$$

If $k' > k$ and $\ell' = 0$ we use (2.28) to get $a(k,\ell,k',\ell') \leq 2^{k-k'-\ell}$.

If $k' > k$ and $\ell' \geq 1$ we use (2.30) and we observe that $n_k^{\ell+2} \leq n_k^{k+2} \leq \sqrt{n_{k'}}$ since $n_{k'} \geq n_{k+1} \geq n_k^{2(k+2)}$, so we get $a(k,\ell,k',\ell') \leq 2^k n_{k'}^{-1/4}$.

If $k' = k$ and $\ell' \geq \ell + 3$ we use (2.30) to see that $a(k,\ell,k',\ell') \leq 2^k n_k^{-1/2}$.

If $k' = k$ and $\ell' \leq \ell + 2$ we use (2.28) to get that $a(k,\ell,k',\ell') \leq L 2^{\ell'-\ell}$.

If $k' < k$ and $\ell' + 1 \leq k - 1$ we use (2.29) and $n_{k'}^{\ell'+1} \leq n_{k-1}^{k-1}$, $c_k = 2^k n_{k-1}^{-k/2}$ to get $a(k,\ell,k',\ell') \leq 2^k n_{k-1}^{-1/2}$.

Finally, if $k < k'$, and $\ell' + 1 > k - 1$, since $\ell' \leq k'$ we have $k' > k - 2$ so that $k' = \ell' = k - 1$. In that case we use (2.14) and (2.23) to get $\|x_{k,\ell}\|_{k-1,k-1} \leq \mathcal{N}_k(x_{k,\ell}) \leq 2^{-\ell}\alpha_{k,\ell}$ and hence $a(k,\ell,k',\ell') \leq 2^{-\ell}$.

It is then immediate to check (2.26) and (2.27). ∎

We define $c'_{k,0} = 2c_{k-1}$ and for $\ell \geq 1$ we define $c'_{k,\ell} = c_{k,\ell} = 2^{k-\ell} n_k^{-\ell/2}$.

LEMMA 2.5: *If $|||\,x\,|||_2 \leq 1$, we can find a decomposition*

$$x = \sum_{k \text{ even}, 0 \leq \ell \leq k} x_{k,\ell}$$

and numbers $\alpha_{k,\ell} \geq 0$ such that the elements $x_{k,\ell}$ have disjoint supports and

$$\sum_{k \text{ even}} \Big(\sum_{0 \leq \ell \leq k} \alpha_{k,\ell}^r \Big)^{2/r} \leq L, \tag{2.31}$$

$$\|x_{k,\ell}\|_2 \leq 2^{k-\ell} \alpha_{k,\ell}, \tag{2.32}$$

$$\|x_{k,\ell}\|_\infty \leq c'_{k,\ell} \alpha_{k,\ell}, \tag{2.33}$$

$$\mathrm{CS}(x_{k,\ell}) \leq n_k^{\ell+1}, \tag{2.34}$$

$$\mathcal{N}_{k-1}(x_{k,0}) \leq \alpha_{k,0}. \tag{2.35}$$

Proof: Without loss of generality we can assume that the sequence $(|x(i)|)$ is non-increasing. We define $\beta_{k,\ell} = \|x\|_{k,\ell}$. For $\ell \geq 1$ we define $\alpha_{k,\ell} = \beta_{k,\ell}$. We define $\alpha_{2,0} = 2\beta_{2,0}$ and for $k \geq 4$ we define $\alpha_{k,0} = \beta_{k,0} + \beta_{k-2,k-2}$. For $k \geq 2$, k even, we define $x_{k,0}(i) = x(i)$ if $n_{k-2}^{k-1} \leq i < n_k$ and $x_{k,0}(i) = 0$ otherwise. We use (2.10) with $s = 1, t = n_k, a = 2^{-k}$ and $b = \beta_{k,0}$ to get $\sum_{i \leq n_k} x(i)^2 \leq 2^{2k}\beta_{k,0}$ and $ix(i)^2 \leq 2^{2k}\beta_{k,0}$, which implies (2.33). When $k \geq 2$ we use (2.10) with $s = n_{k-2}^{k-2}$, $t = n_{k-2}^{k-1}, a = 1$ and $b = \beta_{k-2,k-2}$ to get $\mathcal{N}_{k-1}(x_{k,0}) \leq \beta_{k-2,k-2}$. The rest is as in the case of Lemma 2.3. ∎

The proof of the following is very similar to that of Lemma 2.4 and is left to the reader.

LEMMA 2.6: *Assume that for k even and $0 \leq \ell \leq k$ we are given elements $x_{k,\ell} \in Z$ and numbers $\alpha_{k,\ell} \geq 0$ such that*

$$\|x_{k,\ell}\|_2 \leq 2^{k-\ell} \alpha_{k,\ell}, \tag{2.36}$$

$$\|x_{k,\ell}\|_\infty \leq 2c_{k-1} \alpha_{k,\ell}, \tag{2.37}$$

$$\mathrm{CS}(x_{k,\ell}) \leq n_k^{\ell+2}, \tag{2.38}$$

$$\mathcal{N}_{k-1}(x_{k,\ell}) \leq 2^{-\ell} \alpha_{k,\ell}. \tag{2.39}$$

Then we have

$$\Big|\Big|\Big| \sum_{k \text{ even}} \sum_{0 \leq \ell \leq k} x_{k,\ell} \Big|\Big|\Big|_2 \leq L \Big(\sum_{k \text{ even}} \Big(\sum_{0 \leq \ell \leq k} \alpha_{k,\ell}^r)^{2/r} \Big)^{1/2}. \tag{2.40}$$

Of course, similar results hold for a summation over k odd.

3. The decomposition property

The basic property of our construction is as follows.

THEOREM 3.1: *Consider vectors $(x_j)_{j\le N}$ of X. Then we can find an increasing family $(I_j)_{j\le N}$ of subsets of $\mathbb{N}$ such that*

$$j \le m_{k+1} \Longrightarrow \operatorname{card} I_j \le k2^{2k+1}/c_k^2 \tag{3.1}$$

and such that if $x'_j = x_j 1_{I_j^c}$, we have

$$\Big\| \Big(\sum_{j\le N} x_j'^2 \Big)^{1/2} \Big\| \le L \Big(\sum_{j\le N} \|x_j\|^2 \Big)^{1/2}. \tag{3.2}$$

Proof: To simplify notation we set $x_j = 0$ for $j > N$. All summations over j will involve only finitely non-zero terms. By definition of $\|\cdot\|$, and Lemmas 2.3 and 2.5, for $j \ge 1$ and $k \ge 1$, $0 \le \ell \le k$, we can find numbers $\alpha_{k,\ell,j} \ge 0$ and $\beta_{k,\ell,j} \ge 0$ such that

$$\Big(\sum_{k\ge 1, 0\le \ell \le k} \alpha_{k,\ell,j}^r \Big)^{1/r} \le \|x_j\|, \tag{3.3}$$

$$\Big(\sum_{k \text{ even}} \Big(\sum_{0\le \ell\le k} \beta_{k,\ell,j}^r \Big)^{2/r} \Big)^{1/2} \le \|x_j\|, \tag{3.4}$$

$$\Big(\sum_{k \text{ odd}} \Big(\sum_{0\le \ell\le k} \beta_{k,\ell,j}^r \Big)^{2/r} \Big)^{1/2} \le \|x_j\|, \tag{3.5}$$

and elements $x_{k,\ell,j}, y_{k,\ell,j}$ of Z such that

(3.6) the vectors $(x_{k,\ell,j})$ have disjoint supports for $k \ge 1, 0 \le \ell \le k$,

$$\|x_{k,\ell,j}\|_2 \le 2^{k-\ell} \alpha_{k,\ell,j}, \tag{3.7}$$

$$\|x_{k,\ell,j}\|_\infty \le c_{k,\ell} \alpha_{k,\ell,j}, \tag{3.8}$$

$$\operatorname{CS}(x_{k,\ell,j}) \le n_k^{\ell+1}, \tag{3.9}$$

$$\mathcal{N}_k(x_{k,0,j}) \le \alpha_{k,0,j}, \tag{3.10}$$

(3.11) the vectors $(y_{k,\ell,j})$ have disjoint supports for k even and $0 \le \ell \le k$,

(3.12) the vectors $(y_{k,\ell,j})$ have disjoint supports for k odd and $0 \le \ell \le k$,

$$\|y_{k,\ell,j}\|_2 \le 2^{k-\ell} \beta_{k,\ell,j}, \tag{3.13}$$

$$\|y_{k,\ell,j}\|_\infty \le c'_{k,\ell}\beta_{k,\ell,j}, \tag{3.14}$$

$$\mathrm{CS}(y_{k,\ell,j}) \le n_k^{\ell+1}, \tag{3.15}$$

$$\mathcal{N}_{k-1}(y_{k,0,j}) \le \beta_{k,0,j}, \tag{3.16}$$

$$|x_j| \le \sum_{k\ge 1, 0\le \ell\le k} (|x_{k,\ell,j}| + |y_{k,\ell,j}|). \tag{3.17}$$

Consider for $k \ge 1$ and $0 \le \ell \le k$ the function $v_{k,\ell}$ given by

$$v_{k,\ell}^2 = \sum (x_{k,\ell,j}^2 + y_{p,\ell,j}^2), \tag{3.18}$$

where the summation is over j such that $m_k < j \le m_{k+1}$ and p such that $\max(1,\ell) \le p \le k$.

By Markov's inequality there exists a set $A_{k,\ell}$ with $\operatorname{card} A_{k,\ell} \le 2^{2k-2\ell}/c_k^2$ such that

$$\|v_{k,\ell} 1_{A_{k,\ell}^c}\|_\infty \le 2^{\ell-k} c_k \|v_{k,\ell}\|_2. \tag{3.19}$$

We define

$$I_j = \bigcup_{m_p<j} \bigcup_{\ell\le p} A_{p,\ell}.$$

If $j \le m_{k+1}$ then $m_p < j \Rightarrow p \le k$, and since $c_p \ge c_k$ for $p \le k$ this proves (3.1). Of course the reason for this definition will become clear only when we perform the main calculation, which we start now. We keep the convention that $\sum_{k,\ell}$ means summation for $k \ge 1$ and $0 \le \ell \le k$. Using (3.17) we have

$$x_j^2 1_{I_j^c} \le L \sum_{k,\ell} (x_{k,\ell,j}^2 + y_{k,\ell,j}^2) 1_{I_j^c} \tag{3.20}$$

so that

$$\sum_{j\ge 1} x_j^2 1_{I_j^c} \le L \sum_{k,\ell} \sum_{j\ge 1} (x_{k,\ell,j}^2 + y_{k,\ell,j}^2) 1_{I_j^c}. \tag{3.21}$$

We define $x_{k,\ell} \ge 0$ and $y_{k,\ell} \ge 0$ by

$$x_{k,\ell}^2 = \sum_{j\le m_k} x_{k,\ell,j}^2; \quad y_{k,\ell}^2 = \sum_{j\le m_k} y_{k,\ell,j}^2 \tag{3.22}$$

and for $q \ge k$ we set

$$w_{k,\ell,q} = \sum_{m_q<j\le m_{q+1}} (x_{k,\ell,j}^2 + y_{k,\ell,j}^2) 1_{I_j^c},$$

so that by (3.21) we get

$$\sum_{j\geq 1} x_j^2 1_{I_j^c} \leq L \sum_{k,\ell}\Big(x_{k,\ell}^2 + y_{k,\ell}^2 + \sum_{q\geq k} w_{k,\ell,q}\Big). \tag{3.23}$$

Now, for $j > m_q$ and $\ell \leq q$ we have $I_j \supset A_{q,\ell}$, so that $I_j^c \subset A_{q,\ell}^c$ and

$$\begin{aligned}\sum_{q\geq k\geq \ell\geq 0, k\geq 1} w_{k,\ell,q} &\leq \sum_{q\geq k\geq \ell\geq 0,k\geq 1} 1_{A_{q,\ell}^c} \sum_{m_q<j\leq m_{q+1}} (x_{k,\ell,j}^2 + y_{k,\ell,j}^2)\\ &\leq \sum_{q\geq 1,q\geq \ell\geq 0} 1_{A_{q,\ell}^c} \sum_{m_q<j\leq m_{q+1}} \sum_{k\geq 1, q\geq k\geq \ell} (x_{k,\ell,j}^2 + y_{k,\ell,j}^2)\\ &= \sum_{q\geq 1,q\geq\ell\geq 0} v_{q,\ell}^2 1_{A_{q,\ell}^c}\end{aligned}$$

using (3.18). Thus from (3.23) we see that

$$\Big(\sum_{j\geq 1} x_j^2 1_{I_j^c}\Big)^{1/2} \leq L \sum_{k,\ell} (x_{k,\ell} + y_{k,\ell} + v_{k,\ell} 1_{A_{k,\ell}^c}).$$

To finish the proof we will show that, setting $S = (\sum_{j\geq 1} \|x_j\|^2)^{1/2}$, we have

$$\Big|\Big|\Big| \sum_{k,\ell} x_{k,\ell} \Big|\Big|\Big|_1 \leq LS, \tag{3.24}$$

$$\Big|\Big|\Big| \sum_{k \text{ even},0\leq \ell\leq k} y_{k,\ell} \Big|\Big|\Big|_2 \leq LS, \tag{3.25}$$

$$\Big|\Big|\Big| \sum_{k \text{ odd},0\leq \ell\leq k} y_{k,\ell} \Big|\Big|\Big|_3 \leq LS, \tag{3.26}$$

$$\Big|\Big|\Big| \sum_{k \text{ even},0\leq \ell\leq k} v_{k,\ell} 1_{A_{k,\ell}^c} \Big|\Big|\Big|_3 \leq LS, \tag{3.27}$$

$$\Big|\Big|\Big| \sum_{k \text{ odd},0\leq \ell\leq k} v_{k,\ell} 1_{A_{k,\ell}^c} \Big|\Big|\Big|_2 \leq LS. \tag{3.28}$$

The reader notes the key fact, the use of $|||\cdot|||_3$ in (3.27) and of $|||\cdot|||_2$ in (3.28). To prove (3.24) we set

$$\alpha_{k,\ell} = \Big(\sum_{j\leq m_k} \alpha_{k,\ell,j}^2\Big)^{1/2}$$

and we use (3.7) to (3.9) to see that

$$\|x_{k,\ell}\|_2 \leq 2^{k-\ell}\alpha_{k,\ell},$$

$$\|x_{k,\ell}\|_\infty \le c_{k,\ell}\alpha_{k,\ell},$$
$$\mathrm{CS}(x_{k,\ell}) \le n_k^{\ell+2},$$
$$\mathcal{N}_k(x_{k,\ell}) \le 2^{-\ell}\alpha_{k,\ell}. \tag{3.29}$$

To prove the last relation we observe first the general fact that $\mathcal{N}_k((\sum y_j^2)^{1/2}) \le (\sum \mathcal{N}_k^2(y_j))^{1/2}$, which is obvious if one keeps in mind that $\mathcal{N}_k(y)$ is the supremum of the ℓ^2 norms of certain finite-dimensional projections of y. We then note that $\mathcal{N}_k(x_{k,0,j}) \le \alpha_{k,0,j}$ by (3.10), while for $\ell > 1$ we use (3.8) and the fact that we have $\mathcal{N}_k(x_{k,\ell,j}) \le n_{k-1}^{k/2}\|x_{k,\ell,j}\|_\infty$ and $n_{k-1}^{k/2}c_{k,\ell} \le n_{k-1}^{k/2}n_k^{-1/2} \le 2^{-k} \le 2^{-\ell}$.

Thus from Lemma 2.4 we have

$$\left\|\left\|\sum_{k,\ell} x_{k,\ell}\right\|\right\|_1^2 \le L\Big(\sum_{k,\ell}\alpha_{k,\ell}^r\Big)^{2/r} = L\Big(\sum_{k,\ell}\Big(\sum_j \alpha_{k,\ell,j}^2\Big)^{r/2}\Big)^{2/r} \le LS^2,$$

using (3.3) and the triangle inequality for the norm $\|\cdot\|_{r/2}$.

The proof of (3.25) and (3.26) is entirely similar and we turn to the proof of (3.28). We define $\beta_{k,\ell} \ge 0$ by

$$\beta_{k,\ell}^2 = \sum(\alpha_{p,\ell,j}^2 + \beta_{p,\ell,j}^2)2^{2(p-k)}, \tag{3.30}$$

where the summation is over j with $m_k < j \le m_{k+1}$ and p with $\max(1,\ell) \le p \le k$. From (3.7) and (3.13) we see that

$$\|v_{k,\ell}\|_2^2 \le \sum(\|x_{p,\ell,j}\|_2^2 + \|y_{p,\ell,j}\|_2^2) \le \sum 2^{2(p-\ell)}(\alpha_{p,\ell,j}^2 + \beta_{p,\ell,j}^2) = 2^{2(k-\ell)}\beta_{k,\ell}^2$$

where the summations are as in (3.30), so that

$$\|v_{k,\ell}\|_2 \le 2^{k-\ell}\beta_{k,\ell}.$$

Using this inequality and (3.19) we similarly see that

$$\|v_{k,\ell}1_{A_{k,\ell}^c}\|_\infty \le c_k\beta_{k,\ell}.$$

Moreover, by (3.9) and (3.15) we get

$$\mathrm{CS}(v_{k,\ell}) \le 2m_{k+1}\sum_{p\le k} n_p^{\ell+1} \le 2km_{k+1}n_k^{\ell+1} \le n_{k+1}^{\ell+2}.$$

Writing $\gamma_{p,\ell} = \beta_{p-1,\ell}$ and $w_{p,\ell} = v_{p-1,\ell}1_{A_{p-1,\ell}^c}$ we see that (setting $k = p-1$)

$$\sum_{k \text{ odd}, 0\le\ell\le k} v_{k,\ell}1_{A_{k,\ell}^c} = \sum_{p \text{ even } 0\le\ell\le p-1} w_{p,\ell}$$

where

$$\|w_{p,\ell}\|_2 \le 2^{p-\ell}\gamma_{p,\ell}, \|w_{p,\ell}\|_\infty \le c_{p-1}\gamma_{p,\ell}, \mathrm{CS}(w_{p,\ell}) \le n_p^{\ell+2}, \mathcal{N}_{k-1}(w_{p,\ell}) \le 2^{-\ell}\gamma_{p,\ell},$$

the last relation being obtained as (3.29). Thus from Lemma 2.6 we have

$$\Big\| \sum_{k \text{ odd}, 0\le \ell \le k} v_{k,\ell} 1_{A^c_{k,\ell}} \Big\|_2 \le \Big(\sum_{k \text{ odd}} \Big(\sum_{0\le \ell\le k} \beta^r_{k,\ell} \Big)^{2/r} \Big)^{1/2}. \tag{3.31}$$

We turn to the control of the right-hand side. Let $\gamma_{p,\ell,j} = \alpha_{p,\ell,j} + \beta_{p,\ell,j}$ so that by (3.3), (3.4) and (3.5) we have

$$\forall p \ge 1, \quad \Big(\sum_{0\le \ell\le p} \gamma^r_{p,\ell,j} \Big)^{2/r} \le L\|x_j\|^2, \tag{3.32}$$

and by (3.30) we have

$$\beta^2_{k,\ell} \le \sum \gamma^2_{p,\ell,j} 2^{2(p-k)},$$

where the summation is as in (3.30). Thus, if we set $\delta_{p,\ell,j} = \gamma_{p,\ell,j}$ if $\ell \le p$ and $\delta_{p,\ell,j} = 0$ if $\ell > p$ we get

$$\beta^2_{k,\ell} \le \sum_{m_k < j \le m_{k+1}} \sum_{1\le p\le k} \delta^2_{p,\ell,j} 2^{2(p-k)}.$$

Using the triangle inequality for the norm $\|\cdot\|_{r/2}$, we get that for each k we have

$$\begin{aligned}\Big(\sum_{0\le \ell\le k} \beta^r_{k,\ell} \Big)^{2/r} &\le \sum_{m_k < j \le m_{k+1}} \sum_{1\le p\le k} \Big(\sum_{0\le \ell\le k} \delta^r_{p,\ell,j} 2^{r(p-k)} \Big)^{2/r} \\ &= \sum_{m_k < j \le m_{k+1}} \sum_{1\le p\le k} 2^{2(p-k)} \Big(\sum_{0\le \ell\le p} \gamma^r_{p,\ell,j} \Big)^{2/r} \\ &\le \sum_{m_k < j \le m_{k+1}} \sum_{1\le p\le k} 2^{2(p-k)} \|x_j\|^2,\end{aligned}$$

using (3.32) in the last line. Performing this summation we find that the right-hand side of (3.31) is at most LS. This proves (3.28). The proof of (3.27) is similar. ∎

4. Failure of type 2

We will need the following facts, true in any symmetric sequence space.

PROPOSITION 4.1: (a) *Given elements $(x_j)_{j\le N}$ of X we have*

$$\left\|\left(\sum_{j\le N} x_j^2\right)^{1/2}\right\| \le L\,\mathrm{Av}\left\|\sum_{j\le N}\eta_j x_j\right\|. \tag{4.1}$$

(b) *If for some $r>2$ the space X is of cotype r, then*

$$\mathrm{Av}\left\|\sum_{j\le N}\eta_j x_j\right\| \le C\left\|\left(\sum_{j\le N} x_j^2\right)^{1/2}\right\| \tag{4.2}$$

(where C depends on X only).

Proof: If $y^*\in X^*, \|y^*\|\le 1$, then, by Khintchine's inequality,

$$\begin{aligned} y^*\left(\left(\sum_{j\le N} x_j^2\right)^{1/2}\right) &= \sum_{i\ge 1} y^*(i)\left(\sum_{j\le N} x_j^2(i)\right)^{1/2} \\ &\le L\,\mathrm{Av}\sum_{i\ge 1}|y^*(i)|\left|\sum_{j\le N}\eta_j x_j(i)\right| \\ &\le L\,\mathrm{Av}\left\|\sum_{j\le N}\eta_j x_j\right\| \end{aligned}$$

and this proves (a). We refer the reader to [1] for a proof of (4.2). ∎

From (4.1), to prove that X is not of type 2, it suffices to show that for any number $A>0$ we can find elements $(x_j)_{j\le N}$ of X with

$$\left\|\left(\sum_{j\le N} x_j^2\right)^{1/2}\right\| \ge A\left(\sum_{j\le N}\|x_j\|^2\right)^{1/2}.$$

Consider an integer q and disjoint sets $(I_k)_{k\le q}$ of $\mathbb{N}$, with $\operatorname{card} I_k = n_k$, and

$$x=\sum_{k\le q}\frac{2^k}{\sqrt{n_k}}1_{I_k} := \sum_{k\le q} x_{k,0}.$$

We see from Lemma 2.4 (used for $\alpha_{k,\ell}=0$ if $\ell\ge 1$ and $\alpha_{k,0}=1$ for $k\le q$) that $\|x\|\le Lq^{1/r}$. Consider disjoint sets $(J_k)_{k\le q}$ of $\mathbb{N}$, with $\operatorname{card} J_k = n_k^k$, and

$$x'=\sum_{k\le q}\frac{2^k}{\sqrt{n_k^k}}1_{J_k}.$$

It should be obvious that $x'^2 = N^{-1}\sum_{j\le N} x_j^2$ for a certain N, where x_j is obtained from x by a suitable permutation of the coordinates, so that

$$\left(\sum_{j\le N}\|x_j\|^2\right)^{1/2} \le L\sqrt{N}q^{1/r}. \tag{4.3}$$

Consider

$$y' = \sum_{k\le q}\frac{1}{2^k\sqrt{n_k^k}}1_{J_k} := \sum_{k\le q} y_k$$

so that

$$\langle y', x'\rangle = q$$

and thus

$$\|x'\| \ge \frac{q}{\|y'\|_*}. \tag{4.4}$$

Now, we observe that

$$\|y_k\|_2 \le 2^{-k}, \quad \mathrm{CS}(y_k) \le n_k^k, \quad \|y_k\|_\infty \le 2^{-k}/\sqrt{n_k^k},$$

so that

$$\langle y_k, x\rangle \le L \le 2^{-k}\|x\|_{k,k} \le 2^{-k}\,|||\,x\,|||_1;$$
$$\langle y_k, x\rangle \le L\|x\|_{k+1,0}; \quad \langle y_k, x\rangle \le L\|x\|_{k+2,0}.$$

Thus we have $\langle y', x\rangle \le 2\,|||\,x\,|||_1$ and

$$\langle y', x\rangle \le L\sum_{k\ \text{even}, k\le q+2}\|x\|_{k,0} \le L\sqrt{q}\,|||\,x\,|||_2$$

and similarly $\langle y', x\rangle \le L\sqrt{q}\,|||\,x\,|||_3$, so that $\|y'\|_* \le L\sqrt{q}$. Thus (4.4) shows that $\|x'\| \ge \sqrt{q}/L$ and, comparing with (4.3), this finishes the proof. ∎

It is also good to note the following.

LEMMA 4.1: *Given $A > 0$, we can find elements $(y_j)_{j\le N}$ of X^* with*

$$\left(\sum_{j\le N}\|y_j\|_*^2\right)^{1/2} \ge A\left\|\left(\sum_{j\le N} y_j^2\right)^{1/2}\right\|_*.$$

Proof: With the previous notation, let

$$y = \sum_{k\le q}\frac{1}{2^k\sqrt{n_k}}1_{I_k}$$

so $\langle x,y\rangle = q$ and $\|y\|_* \geq Lq^{1-1/r}$ since $\|x\| \leq Lq^{1/r}$. Obviously we have $y'^2 = N^{-1}\sum_{j\leq N} y_j^2$ where y_j is obtained from y by a suitable permutation of the coordinates, and we have shown that $\|y'\|_* \leq L\sqrt{q}$. ∎

5. Study of X^*

In this section we assume that the sequence m_k grows fast enough to ensure that $m_k \geq c_k^{-4}2^{5k}$ (the crucial point being that c_k depends only on m_{k-1}).

THEOREM 5.1: *The space X^* satisfies the Orlicz property (i.e., is of sup-cotype 2), but is not of cotype 2.*

The following property will be crucial.

PROPOSITION 5.2: *Consider vectors $(y_j)_{j\leq N}$ in X^*. Then we can find an increasing family $(I_j)_{j\leq N}$ of subsets of $\mathbb{N}$ with*

$$j \leq m_{k+1} \Longrightarrow \operatorname{card} I_j \leq k2^{2k+1}/c_k^2 \tag{5.1}$$

and such that

$$\Big(\sum_{j\leq N}\|y_j\|_*^2\Big)^{1/2} \leq L\bigg(\Big\|\Big(\sum_{j\leq N} y_j^2\Big)^{1/2}\Big\|_* + \Big(\sum_{j\leq N}\|y_j 1_{I_j}\|_*^2\Big)^{1/2}\bigg). \tag{5.2}$$

Proof: For $j \leq N$ we consider $x_j \in X$, with $\|x_j\| = \|y_j\|_*$ and $\langle y_j, x_j\rangle \geq \|y_j\|_*^2/2$. Then, by Theorem 3.1 we can find sets I_j as in (5.1) such that

$$\Big\|\Big(\sum_{j\leq N} x_j^2 1_{I_j^c}\Big)^{1/2}\Big\| \leq L\Big(\sum_{j\leq N}\|y_j\|_*^2\Big)^{1/2}.$$

Using the Cauchy-Schwartz inequality on each coordinate we have

$$\begin{aligned}\sum_{j\leq N}\langle y_j, x_j 1_{I_j^c}\rangle &\leq \bigg\langle\Big(\sum_{j\leq N} y_j^2\Big)^{1/2}, \Big(\sum_{j\leq N} x_j^2 1_{I_j^c}\Big)^{1/2}\bigg\rangle \\ &\leq \Big\|\Big(\sum_{j\leq N} y_j^2\Big)^{1/2}\Big\|_*\Big\|\Big(\sum_{j\leq N} x_j^2 1_{I_j^c}\Big)^{1/2}\Big\| \\ &\leq L\Big\|\Big(\sum_{j\leq N} y_j^2\Big)^{1/2}\Big\|_*\Big(\sum_{j\leq N}\|y_j\|_*^2\Big)^{1/2}.\end{aligned} \tag{5.3}$$

On the other hand, we have

$$\begin{aligned}\sum_{j\le N}\langle y_j, x_j 1_{I_j^c}\rangle &\ge \sum_{j\le N}(\langle y_j,x_j\rangle - \langle y_j, x_j 1_{I_j}\rangle)\\ &\ge \frac12\sum_{j\le N}\|y_j\|_*^2 - \sum_{j\le N}\langle y_j 1_{I_j}, x_j\rangle\\ &\ge \frac12\sum_{j\le N}\|y_j\|_*^2 - \sum_{j\le N}\|y_j 1_{I_j}\|_*\|y_j\|_*\\ &\ge \frac12\sum_{j\le N}\|y_j\|_*^2 - \Big(\sum_{j\le N}\|y_j 1_{I_j}\|_*^2\Big)^{1/2}\Big(\sum_{j\le N}\|y_j\|_*^2\Big)^{1/2}.\end{aligned}$$

Combining with (5.3) yields the result. ∎

Proof of Theorem 5.1: We first prove the Orlicz property. Consider vectors $(y_j)_{j\le N}$ of X^* and assume that for all signs $(\eta_j)_{j\le N}$ we have

$$\Big\|\sum_{j\le N}\eta_j y_j\Big\|_* \le 1. \tag{5.4}$$

Then $\mathrm{Av}\Big\|\sum_{j\le N}\eta_j y_j\Big\|_* \le 1$, and, as shown by Proposition 4.1, we have

$$\Big\|\Big(\sum_{j\le N} y_j^2\Big)^{1/2}\Big\|_* \le L. \tag{5.5}$$

It should be obvious that $\|y\|_* \ge |y(i)|/L$ for each i, so by (5.4) we have

$$\forall i,\quad \sum_{j\le N}|y_j(i)| \le L,$$

and hence, for any set I, we have

$$\sum_{j\le N, i\in I}|y_j(i)| \le L\,\mathrm{card}\, I. \tag{5.6}$$

Since it is easily seen that $\|x\|\le 1 \Longrightarrow |x(i)|\le L$, for all i we have

$$\|y1_I\|_* \le L\sum_{i\in I}|y(i)|$$

and (5.6) shows that

$$\sum_{j\le N}\|y_j 1_I\|_* \le L\,\mathrm{card}\, I. \tag{5.7}$$

Let $S^2 = \sum_{j\le N} \|y_j\|_*^2$. We can assume without loss of generality that the sequence $(\|y_j\|_*^2)$ is non-increasing so that $\|y_j\|_*^2 \le S^2/j$. Thus

$$\sum_{m_k<j\le m_{k+1}} \|y_j 1_{I_j}\|_*^2 \le \frac{S}{\sqrt{m_k}} \sum_{j\le m_{k+1}} \|y_j 1_{I_j}\|_* \le LS\,\frac{k2^{2k+1}}{c_k^2\sqrt{m_k}}$$

using (5.1), (5.7) and the fact that $\operatorname{card} I_j \le \operatorname{card} I_{m_{k+1}}$ for $j\le m_{k+1}$. Since we assume $m_k \ge c_k^{-4}2^{5k}$, we get $\sum_{j\le N}\|y_j 1_{I_j}\|_*^2 \le LS$, and (5.2) yields $S\le L(1+\sqrt{S})$, so $S\le L$. This proves the Orlicz property.

Since X^* satisfies the Orlicz property, it is of cotype q for each $q>2$, and hence not of cotype 2 by Proposition 4.1 and Lemma 4.1. ∎

6. Cotype r

Theorem 6.1: *The space X is of cotype r.*

Proof: Using (4.1) it suffices to show that for vectors $(x_j)_{j\le N}$ in X we have

$$\left(\sum_{j\le N}\|x_j\|^r\right)^{1/r} \le L\left\|\left(\sum_{j\le N} x_j^2\right)^{1/2}\right\|. \tag{6.1}$$

By homogeneity we can assume $\|(\sum_{j\le N} x_j^2)^{1/2}\| = 1$. By definition of $\|\cdot\|$, for $q=1,2,3$ we can find y_q with $|||\,y_q\,|||_q \le 1$ and $(\sum_{j\le N} x_j^2)^{1/2} \le y_1+y_2+y_3$, so that

$$\sum_{j\le N} x_j^2 \le 3(y_1^2+y_2^2+y_3^2). \tag{6.2}$$

For $q\le 3$, let $I_q = \{i\in\mathbb{N}; y_q(i) = \max_{\ell\le 3}|y_\ell(i)|\}$, so that

$$\sum_{j\le N} x_j^2 \le L\sum_{q\le 3} y_q^2 1_{I_q}.$$

We will show that for $q\le 3$ we have

$$\sum_{j\le N} |||\,x_j 1_{I_q}\,|||_q^r \le L.$$

We assume first $q=1$. We apply Lemma 2.3 to $x=y_1$. Let $I_{k,\ell}$ be the support of $x_{k,\ell}$ so that, if $x_{j,k,\ell} = x_j 1_{I_{k,\ell}\cap I_1}$, we have

$$\|x_j 1_{I_1}\|_{k',\ell'} \le \sum_{k,\ell}\|x_{j,k,\ell}\|_{k',\ell'}.$$

Given numbers $d(k,\ell,k',\ell') > 0$ with

$$\forall k',\ell', \quad \sum_{k,\ell} d(k,\ell,k',\ell') \le 1, \tag{6.3}$$

and writing

$$\sum_{k,\ell} \|x_{j,k,\ell}\|_{k',\ell'} = \sum_{k,\ell} d(k,\ell,k',\ell')(d(k,\ell,k',\ell')^{-1}\|x_{j,k,\ell}\|_{k',\ell'}),$$

the convexity of the function $t \mapsto t^r$ shows that

$$\|x_j 1_{I_1}\|_{k',\ell'}^r \le \sum_{k,\ell} d(k,\ell,k',\ell')^{-r+1}\|x_{j,k,\ell}\|_{k',\ell'}^r. \tag{6.4}$$

Now we observe that on the support of $x_{j,k,\ell}$ we have

$$x_{j,k,\ell} \le \Big(\sum_{j\le N} x_{j,k,\ell}^2\Big)^{1/2} \le y_1 + y_2 + y_3 \le 3y_1 = 3x_{k,\ell},$$

so that

$$\|x_{j,k,\ell}\|_{k',\ell'} \le 3\|x_{k,\ell}\|_{k',\ell'}; \quad \sum_{j\le N} x_{j,k,\ell}^2 \le 9x_{k,\ell}^2. \tag{6.5}$$

Combining with (6.4) we get

(6.6)

$$\sum_{j\le N} |||\, x_j 1_{I_1}\, |||_1^r \le L \sum_{k,\ell,k',\ell'} d(k,\ell,k',\ell')^{-r+1}\|x_{k,\ell}\|_{k',\ell'}^{r-2}\Big(\sum_{j\le N}\|x_{j,k,\ell}\|_{k',\ell'}^2\Big).$$

Using (2.16), and since $\sum_{j\le N} x_{j,k,\ell}^2 \le 9x_{k,\ell}^2$ by the second part of (6.5) we see from (2.6) that

$$\sum_{j\le N} \|x_{j,k,\ell}\|_{k',\ell'}^2 \le L\alpha_{k,\ell}^2 2^{2(k-\ell-k'+\ell')}. \tag{6.7}$$

Taking $d(k,\ell,k',\ell')^{-r} = C2^{|k-k'|+|\ell-\ell'|}$ where C is the smallest possible such that (6.3) holds, we see from (6.6) that

$$\sum_{j\le N} |||\, x_j 1_{I_1}\, |||_1^r \le L \sum_{k,\ell,k',\ell'} 2^{|k-k'|+|\ell-\ell'|+2(k-k'+\ell'-\ell)}\alpha_{k,\ell}^2\|x_{k,\ell}\|_{k',\ell'}^{r-2}. \tag{6.8}$$

To conclude the argument, we will use (2.15), and we will show that we can find numbers $a'(k,\ell,k',\ell')$ such that

$$\|x_{k,\ell}\|_{k',\ell'} \le \alpha_{k,\ell} a'(k,\ell,k',\ell'), \tag{6.9}$$

and

$$\forall k,\ell,\quad \sum_{k',\ell'} 2^{|k-k'|+|\ell-\ell'|+2(k-k'+\ell'-\ell)} a'(k,\ell,k',\ell')^{r-2} \le L. \tag{6.10}$$

These numbers are found in a manner similar to the estimates of Lemma 2.4, using now the full strength of (2.17) rather than only (2.21). Specifically, besides the estimates of Lemma 2.4, we use the following ones.

If $k' = k$ and $\ell' \le \ell - 2$ we have $\ell \ge 1$, so $c_{k,\ell} = 2^{k-\ell} n_k^{-\ell/2}$, and we use (2.29) to get that $a'(k,\ell,k',\ell') \le 2^{2k} n_k^{-1/2}$.

If $k' < k$ and $\ell \ge 1$ we use (2.29) and $c_{k,\ell} \le 2^k n_k^{-1/2}, n_{k'}^{\ell'+1} \le n_{k-1}^{k+1} \le n_k^{1/2}$ to get $a'(k,\ell,k',\ell') \le 2^{2k} n_k^{-1/4}$.

(The numbers $a'(k,\ell,k',\ell')$ are very small unless $k = k'$ and $|\ell - \ell'| \le 2$ or $k' = k-1$ and $\ell = 0$ or $k' > k$ and $\ell' = 1$. Only the last case could be dangerous, but this is not the case thanks to the term $2(k-k')$ in the exponent of (6.10).)

This concludes the argument when $q = 1$. Since the cases $q = 2$ and $q = 3$ are similar we treat only the case $q = 2$. To lighten notation we make the convention that all summations are over k and k' even, without mentioning it explicitly. We apply Lemma 2.5 to $x = y_2$ and we denote by $I_{k,\ell}$ the support of $x_{k,\ell}$ so that, if $x_{j,k,\ell} = x_j 1_{I_{k,\ell}\cap I_2}$, we have

$$\|x_j 1_{I_2}\|_{k',\ell'} \le \sum_{k,\ell} \|x_{j,k,\ell}\|_{k',\ell'}. \tag{6.11}$$

By the triangle inequality in the space $\ell^{r/2}$, for numbers $a_{j,k'}$ we have

$$\Big(\sum_{j\le N}\Big(\sum_{k'} a_{j,k'}\Big)^{r/2}\Big)^{2/r} \le \sum_{k'}\Big(\sum_{j\le N} a_{j,k'}^{r/2}\Big)^{2/r}.$$

Using this for

$$a_{j,k'} = \Big(\sum_{\ell'} \|x_j 1_{I_2}\|_{k',\ell'}^r\Big)^{2/r}$$

we get

$$\Big(\sum_{j\le N} |||\, x_j 1_{I_2}\, |||_2^r\Big)^{2/r} \le \sum_{k'}\Big(\sum_{j,\ell'} \|x_j 1_{I_2}\|_{k',\ell'}^r\Big)^{2/r}. \tag{6.12}$$

Using the first part of (6.5) and (6.7) (that remains true in the present case with the same proof) we have

$$\sum_{j\le N} \|x_{j,k,\ell}\|_{k',\ell'}^r \le L\alpha_{k,\ell}^2 2^{2(k-k'+\ell'-\ell)} \|x_{k,\ell}\|_{k',\ell'}^{r-2}. \tag{6.13}$$

Consider numbers $d(k,\ell,k',\ell')$ as in (6.3). Starting from (6.11), proceeding as in (6.4) and using (6.13) we get

$$\sum_{j\le N}\|x_j 1_{I_2}\|_{k',\ell'}^r \le \sum_{k,\ell} d(k,\ell,k',\ell')^{-r+1}\alpha_{k,\ell}^2 2^{2(k-k'+\ell'-\ell)}\|x_{k,\ell}\|_{k',\ell'}^{r-2}.$$

Combining with (6.12) we get

(6.14)

$$\Big(\sum_{j\le N}|||\,x_j 1_{I_2}\,|||_2^r\Big)^{2/r} \le L\sum_{k'}\Big(\sum_{\ell'\le k'}\sum_{k,\ell} d(k,\ell,k',\ell')^{-r+1}\alpha_{k,\ell}^2 2^{2(k-k'+\ell'-\ell)}\|x_{k,\ell}\|_{k',\ell'}^{r-2}\Big)^{2/r}.$$

Let us choose the numbers $d(k,\ell,k',\ell')$ as in the case $q=1$ and let us consider numbers $b(k,\ell,k',\ell')$ such that

$$\forall k,\ell,k',\ell',\quad \|x_{k,\ell}\|_{k',\ell'} \le \alpha_{k,\ell}b(k,\ell,k',\ell'). \tag{6.15}$$

Then from (6.14) we have

$$\Big(\sum_{j\le N}|||\,x_j 1_{I_2}\,|||_2^r\Big)^{2/r} \le L\sum_{k'}\Big(\sum_{\ell'\le k'}\sum_{k,\ell} c(k,\ell,k',\ell')\alpha_{k,\ell}^r\Big)^{2/r} \tag{6.16}$$

where

$$c(k,\ell,k',\ell') = 2^{|k-k'|+|\ell-\ell'|+2(k-k'+\ell'-\ell)}b(k,\ell,k',\ell')^{r-2}.$$

Thus

$$\Big(\sum_{j\le N}|||\,x_j 1_{I_2}\,|||_2^r\Big)^{2/r} \le L\sum_{k'}\Big(\sum_{k} f(k,k')A_k\Big)^{2/r} \tag{6.17}$$

where

$$A_k = \sum_{\ell\le k}\alpha_{k,\ell}^r,$$

$$f(k,k') = \sup_{\ell\le k}\sum_{\ell'\le k'} c(k,\ell,k',\ell').$$

By (2.32) we have

$$\sum_k A_k^{2/r} \le L \tag{6.18}$$

and since $2/r \le 1$ we have

$$\sum_{k'} \Big(\sum_k f(k,k') A_k \Big)^{2/r} \le \sum_{k,k'} f(k,k')^{2/r} A_k^{2/r}.$$

Using (6.17) and (6.18) it suffices to show that the numbers $b(k,\ell,k',\ell')$ can be chosen so that

$$\sum_{k'} f(k,k')^{2/r} \le L.$$

This follows from essentially the same estimates as in the case $q = 1$. ∎

7. Infratype 2

When trying to prove the infratype 2 property one must first find a strategy to construct suitable choices of signs. The simple strategy we will use is made apparent in the following lemma.

LEMMA 7.1: *Given an integer M and a number A, there exists a number $K(M,A)$ with the following property. Consider a Banach space F of dimension M, and vectors $(x_j)_{j\le N}$ in F. Then we can find a partition of $\{1,\dots,N\}$ into 3 sets I_1, I_2, I_3 with $\operatorname{card} I_2 = \operatorname{card} I_3$ and a one-to-one map φ: $I_2 \to I_3$ such that*

(7.1)

$$\operatorname{Av}\Big\| \sum_{j\in I_1} \eta_j x_j + \sum_{j\in I_2} \eta_j (x_j - x_{\varphi(j)}) \Big\| \le \frac{1}{A} \Big(\sum_{j\le N} \|x_j\|^2 \Big)^{1/2} + K(M,A) \max_{j\le N} \|x_j\|.$$

Proof: We consider a number $\varepsilon > 0$, and we partition F into the "shells"

$$C_k = \{x \in F; (1+\varepsilon)^k \le \|x\| < (1+\varepsilon)^{k+1}\}$$

for $k \in \mathbb{Z}$. We then partition each set C_k into $K(M)$ sets U such that $\operatorname{diam} U \le 2\varepsilon(1+\varepsilon)^{k+1}$.

In this manner we have partitioned F in small cells such that if x, y belong to the same cell, we have

$$\|x - y\| \le 10\varepsilon \|x\|. \tag{7.2}$$

To construct the sets I_1, I_2, I_3 we group by pairs as many elements as possible in each cell. Thus x_j and $x_{\varphi(j)}$ belong to the same cell, while each cell contains at most one element $x_j, j \in I_1$, so that

$$\sum_{j\in I_1} \|x_j\| \le K(M,\varepsilon) \max_{j\le N} \|x_j\|.$$

On the other hand, if C denotes the cotype-2 constant of F, we have, using (7.2),

$$\mathrm{Av}\Big\|\sum_{j\in I_2}\eta_j(x_j - x_{\varphi(j)})\Big\| \le C\Big(\sum_{j\in I_2}\|x_j - x_{\varphi(j)}\|^2\Big)^{1/2} \le 10\varepsilon C\Big(\sum_{j\le N}\|x_j\|^2\Big)^{1/2}.$$

We then take $\varepsilon = 1/(10AC)$, and use that C is bounded by a function of M. ∎

THEOREM 7.1: *If the sequence $(m_k)_{k\ge 1}$ increases fast enough, the space X is of infratype 2.*

Proof: Consider elements $(x_j)_{j\ge 1}$ of X, and assume without loss of generality that $\sum_{j\ge 1}\|x_j\|^2 \le 1$ and that the sequence $(\|x_j\|)_{j\ge 1}$ is non-increasing so that $\|x_j\| \le 1/\sqrt{j}$. Let us recall the sets I_j of Theorem 3.1, and set $J_k = I_{k+1}$, so that $M_k = \operatorname{card} J_k \le k2^{2k+1}/c_k^2$ and

$$j \le m_{k+1} \Rightarrow I_j \subset J_k \Rightarrow J_k^c \subset I_j^c.$$

Hence if we set $x'_j = x_j 1_{J_k^c}$ for $m_k < j \le m_{k+1}$, by (3.2) we have

$$\Big\|\Big(\sum_{j\ge 1} x_j'^2\Big)^{1/2}\Big\| \le L. \tag{7.3}$$

Let us write $x''_j = x_j 1_{J_k}$, so that for $m_k < j \le m_{k+1}$, all these vectors belong to a space F_k of dimension M_k. By Lemma 7.1 (used for $A = 2^k$), we can find a partition of $\{m_k+1,\dots,m_{k+1}\}$ into three sets I_1, I_2, I_3 and a one-to-one map φ: $I_2 \to I_3$ such that

$$\mathrm{Av}\Big\|\sum_{j\in I_1}\eta_j x''_j + \sum_{j\in I_2}\eta_j(x''_j - x''_{\varphi(j)})\Big\| \le 2^{-k} + K(M_k, 2^{-k})m_k^{-1/2},$$

using that $\|x''_j\| \le \|x_j\| \le 1/\sqrt{m_k}$. In this manner we construct a partition of $\{1,\dots,N\}$ into three sets I_1, I_2, I_3 and a one-to-one map φ: $I_2 \to I_3$ such that

$$\mathrm{Av}\Big\|\sum_{j\in I_1}\eta_j x''_j + \sum_{j\in I_2}\eta_j(x''_j - x''_{\varphi(j)})\Big\| \le \sum_{k\ge 1}(2^{-k} + K(M_k, 2^{-k})m_k^{-1/2}). \tag{7.4}$$

The number $K(M_k, 2^{-k})$ depends on n_{k-1} only, so that if the sequence (m_k) increases fast enough the right-hand side of (7.4) is finite.

Since X is of cotype r, by Proposition 4.1 and (7.3) we have

$$\mathrm{Av}\left\| \sum_{j\in I_1} \eta_j x'_j + \sum_{j\in I_2} \eta_j (x'_j - x'_{\varphi(j)}) \right\| \le L,$$

and the result follows since $x_j = x'_j + x''_j$. ∎

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